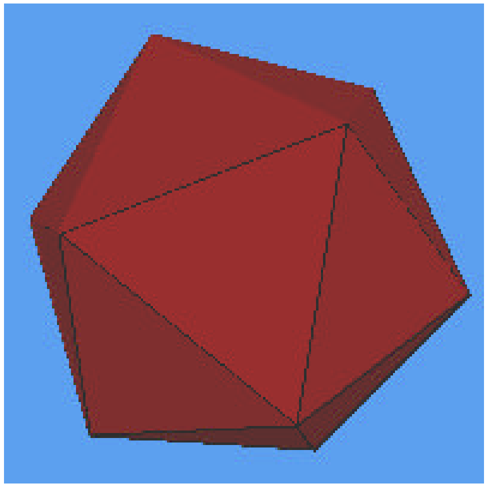
Polyhedral Computation

by Harvey Greenberg, CU-Denver



H-representation:
$\{x: Ax \le b\}$

V-representation:
$\text{convh}\{v_1, \ldots, v_p\}$

$[+ \text{convh}\{(r_1), \ldots, (r_q)\}]$

August 2003

Transforms H $\rightleftarrows$ V

- H $\rightarrow$ V is the vertex enumeration problem (often embedded in face enumeration)
- V $\rightarrow$ H is a form of the convex hull problem (Quickhull algorithm is another form)

Convex Hulls

- Easy in 2-space (Graham scan; Jarvis march)
- $O(p^{\lceil n/2 \rceil})$ in general (maybe just for simplicial polyhedra - i.e., every face is a simplex)

Don't explain why it can't be done.
Discover how it can be done.
Mo Tao (404-319 B.C.)

Enumeration of Extreme Points and Extreme Rays

- Double Description Method, based on Fourier-Motzkin elimination
  - ccd, Fukuda; dda, Padberg
- Reverse Search Algorithm, uses simplex method with systematic search over sequence of bases
  - lrs, Aris
- All implementations are very limited - about 20 variables (maybe 30 max)
- We’ll consider “column generation” to avoid explicit enumeration while satisfying some criteria

Conversion to *Standard form* to use Simplex Method

H-representation

$Ax \leq b$

$\leftrightarrow Ax + s = b,\ s \geq 0$

$\leftrightarrow Au - Av + s = b,\ (u, v, s) \geq 0$

$$[A\ {-A}\ \ I]\begin{bmatrix} u \\ v \\ s \end{bmatrix} = b$$

rank = m

Standard form

$Ax = b,\ x \geq 0,\ \text{rank}(A)=m$

Extreme point x

$\leftrightarrow$ Basic Feasible Solution

$B \subseteq \{1, \ldots, n\}$ for which $|B|=m$,

$A_B=\{A_j\}_{j\in B}$ are linearly independent,

and $[A_B]^{-1}b \geq 0$

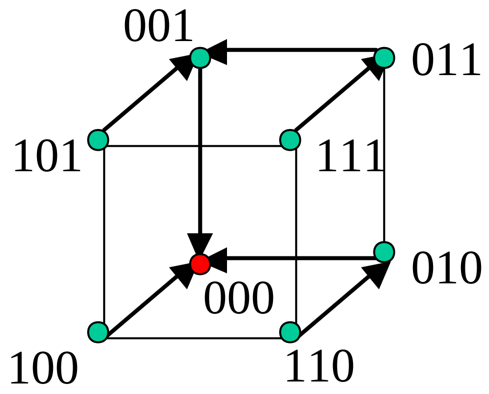
Systematically enumerate all BFSs

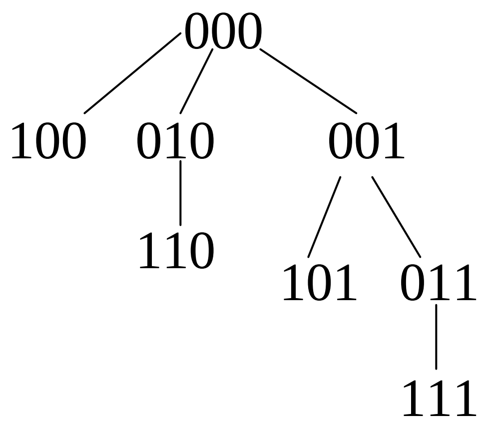
Glitch: extreme point in enlarged space (u, v, s) is not necessarily extreme point in original space (x) (but pathology does not apply to bounded P).

Reverse Search

Basic idea:

1. choose c so that 0 = argmin{cx: Ax ≤ 1, x ≥ 0} (unique)
2. build search tree and reverse the pivots in the simplex method





c = (1, 1, …, 1) ⇒ simplex pivot/search tree induced

ref.: Bremmer, et al. [1998]

Bound on Number of Extreme Points

In standard form, $|\text{ext}(P)| \leq \binom{n}{m}$

m=1: $a(1)x(1) + a(2)x(2) + \ldots + a(n)x(n) = b, \; x \geq 0$

To be bounded, need $a > 0$ or $a < 0$ (no mixed signs)

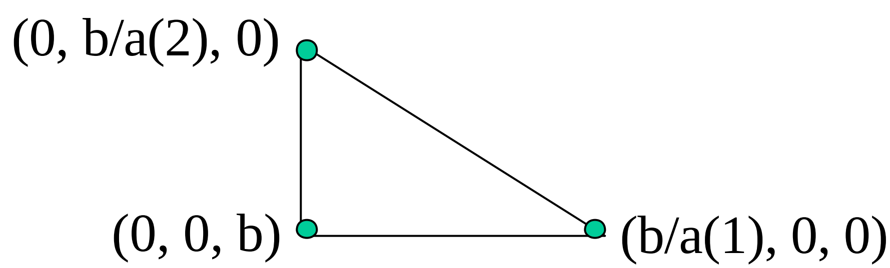
To be full dimensional, need $b \neq 0$

$\text{ext}(P) = \{(b/a(1), 0, \ldots, 0), \; (0, b/a(2), 0, \ldots, 0), \ldots, (0, \ldots, 0, b/a(n))\}$

So, $|\text{ext}(P)| = n$ (i.e., bound is tight for m=1)

$a(1)x(1) + a(2)x(2) + x(3) = b$

$(a(1)x(1) + a(2)x(2) \leq b)$



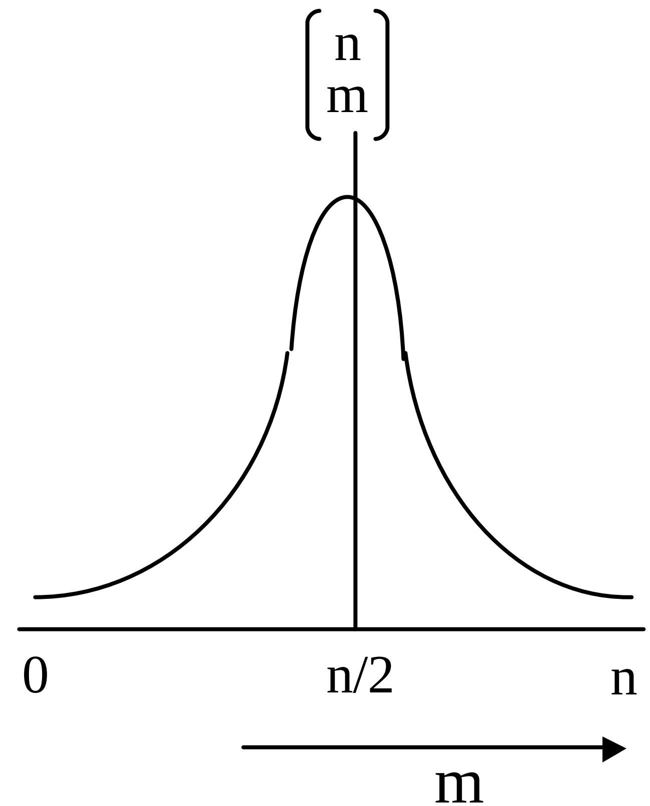
Factorials grow exponentially

In standard form, $|\text{ext}(P)| \leq \binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

Stirling's approximation: $n! \approx \sqrt{2\pi}\, n^{n+½} e^{-n}$

Low end - $m \leq 10\%\, n$: $\binom{n}{m} \approx c \dfrac{(1+r)^n}{\sqrt{n}}$: $r < ½$

High end - $m \approx n/2$: $\binom{n}{m} = \dfrac{2^{n+1}}{\sqrt{2\pi n}}$



Numbers are huge

In standard form, $|\text{ext}(P)| \leq \binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

m fixed: $\binom{n}{m} \approx \dfrac{n^{n+½}}{m!(n–m)^{n–m+½}\, e^{m}} = O(n^{m})$

n	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$
20	1.6×10^{4}	3.8×10^{4}	7.8×10^{4}	1.3×10^{5}	1.7×10^{5}
30	1.4×10^{5}	5.9×10^{5}	2.0×10^{6}	5.9×10^{6}	1.4×10^{7}
40	6.6×10^{5}	3.8×10^{6}	1.9×10^{7}	7.7×10^{7}	2.7×10^{8}
50	2.1×10^{6}	1.6×10^{7}	1.0×10^{8}	5.4×10^{8}	2.5×10^{9}
100	7.5×10^{7}	1.2×10^{9}	1.6×10^{10}	1.9×10^{11}	1.9×10^{12}

100	7.5×10^{8}	1.2×10^{10}	1.6×10^{10}	1.9×10^{11}	1.9×10^{12}
200	2.5×10^{9}	8.2×10^{11}	2.3×10^{13}	5.5×10^{14}	1.2×10^{15}

order of magnitude not reliable at low end

Numbers are huge

In standard form, $|\text{ext}(P)| \leq \binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

Low end - $m \leq 10\%n$: $\binom{n}{m} \approx c \dfrac{(1+r)^n}{\sqrt{n}}$: $r < ½$

n	m = 10% n	$\binom{n}{m}$
20	2	190
30	3	4,060
40	4	91,390
50	5	2.11×10^6
100	10	1.73×10^{13}
200	20	1.61×10^{27}

factoid: postulated age of universe = 10^{17} seconds

Numbers are huge

In standard form, $|\text{ext}(P)| \leq \binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

Latest *E. coli* network: $\binom{\mathbf{931}}{\mathbf{626}}$ **» 1.5 ´ 10^{254}**

High end - $m \approx n/2$: $\binom{n}{m} = \dfrac{2^{n+1}}{\sqrt{2\pi n}}$

n	$\binom{n}{n/2}$
20	1.8×10^{5}
30	1.6×10^{8}
40	1.4×10^{11}
50	1.3×10^{14}
100	1.0×10^{29}
200	9.0×10^{58}

Bounds are not Counts

What portion of the possible extreme points are in fact present?
How can we find out?

I've been so thoroughly trained that I don't even have to think before I speak.

Finding extreme points in V-representation is easy problem

Q: Is v_s and extreme point of convh$\{v_1, \ldots, v_p\}$?

A: No iff $0 = \min\{w_s\colon\; w \geq 0,\; \Sigma_i w_i = 1,\; v_s = \Sigma_i w_i v_i\}$.

$w = e_s$ is feasible

$= (0, \ldots, 0, 1, 0, \ldots, 0)$
↑
s coordinate

$w_s = 0$ means we have $v_s = \Sigma_{i \neq s} w_i v_i$
$\leftrightarrow v_s \in$ convh$\{v_1, \ldots, v_{s-1}, v_{s+1}, \ldots, v_p\}$

Inclusion questions

Is x in P?

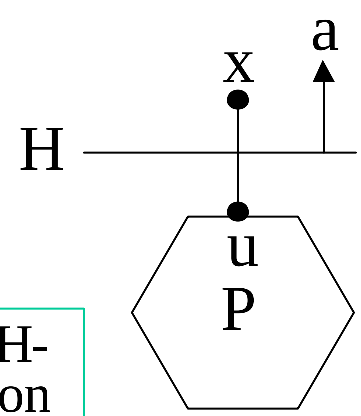
- H-representation: Compute Ax and compare with b
- V-representation: Is x in convh$\{v_j\}$?

  min w_0: $x = \Sigma_j w_j v_j + w_0 x$: $w \geq 0$, $\Sigma_{j=0} w_j = 1$

  $= 0 \leftrightarrow$ yes

- Extension: If no, give separating hyperplane

  min $\Sigma_j w_j$: $w \geq x-u$, $w \geq u-x$, u in P



$w_j = |x_j - u_j|$ at min

'u in P' easy for H- or V-representation

$a = (x-u)$; $H=\{v: av=a(x+u)/2\}$

Inclusion questions

Is $P \cap Q = \varphi$? Let $P = \mathrm{convh}\{v_j\}$ and $Q = \{x: Ax \le b\}$

– $\min y_0: x = \Sigma_j\, y_j v_j + y_0 x:\ y \ge 0,\ \Sigma_{j=0}\, y_j = 1,\ Ax \le b$

$= 0 \leftrightarrow$ no (x is in $P \cap Q$)

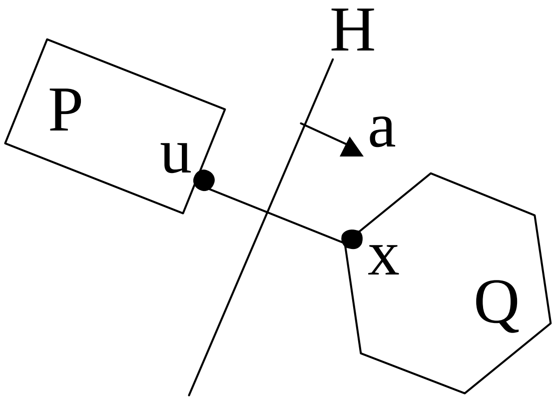
– Extension: If so, give separating hyperplane

$\min \Sigma_j\, w_j :\ w \ge x-u,\ w \ge u-x,$ u in P, x in Q

$w_j = |x_j - u_j|$ at min

simply linear constraints

$a = (x-u);\ H = \{v: av = a(x+u)/2\}$



Volume Computation

- Exact formula
- Simplicial subdivision
- Monte Carlo
- Heuristics

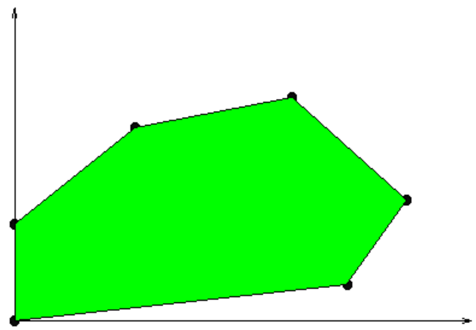
I thought this was new - hadn’t found in literaure; thanks to Steve Bell for pointing to Lovász’s paper (added to refs)

Exact Formula for Polytope

Assume $P=\{x: Ax \leq b\}$ is *simple*

– i.e., $|\{i: A(i, \bullet)x = b(i)\}| = n$ for all $x \in ext(P)$

Assume $0 \in ext(P)$ and $P \subset R^+$



$$vol(P) = \Sigma_{v\in ext(P)} N(v)$$

Let $f(x) = c'x + d$ such that f is non-constant on each edge of P

for each $v \in ext(P)$,

$S = \{i: A(i, \bullet)x = b(i)\}$

$D = |det(A_S)|$

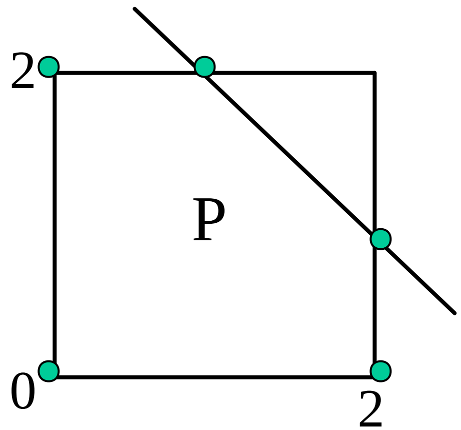
(can update with pivots)

$w = [A_S]^{-1}c$

(i.e., $c = w_1 A_{j_1} + \ldots + w_n A_{j_n}$)

$$N(v) = \frac{f(v)^n}{n!Dw_1 \cdots w_n}$$

Example



$$
\begin{aligned}
-x1 \quad\quad\; &\le 0 \\
-x2 &\le 0 \\
x1 \quad\quad\; &\le 2 \\
x2 &\le 2 \\
x1 + x2 &\le 3
\end{aligned}
$$

ext(P) = {(0, 0),
(2, 0),
(2, 1),
(1, 2),
(0, 2)}

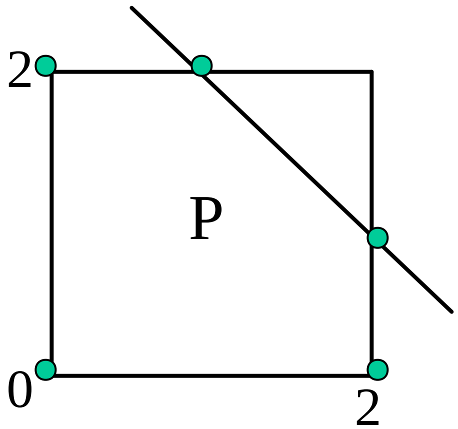
$$c' = (1, -1),\ \ d=0 \Rightarrow f(x) = x1 - x2$$

$$v1 = (0, 0):\ \ A_{S1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ D=1,\ w=(-1, 1)' \Rightarrow N(v1) = 0$$

$$v2 = (2, 0):\ \ A_{S2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ D=1,\ w=(\ 1, 1)' \Rightarrow N(v2) = \frac{2^2}{2!\ 1\cdot 1} = 2$$

$$v3 = (2, 1):\ \ A_{S3} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ D=1,\ w=(2, -1)' \Rightarrow N(v3) = \frac{1^2}{2!\ 2\cdot -1} = -\tfrac{1}{4}$$

Example (con't)



$$
\begin{aligned}
-x1 \quad &\le 0 \\
-x2 &\le 0 \\
x1 \quad &\le 2 \\
x2 &\le 2 \\
x1 + x2 &\le 3
\end{aligned}
$$

ext(P) = {(0, 0), (2, 0), (2, 1), (1, 2), (0, 2)}

$v4 = (1, 2)$: $A_{S4} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, D=1, w=(-2, 1)′ $\Rightarrow N(v4) = \dfrac{1^2}{2!\ -2 \cdot 1} = -¼$

$v5 = (0, 2)$: $A_{S5} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, D=1, w=(-1,-1)′ $\Rightarrow N(v5) = \dfrac{2^2}{2!\ -1 \cdot -1} = 2$

vol(P) = 0 + 2 – ¼ – ¼ + 2 = 3½

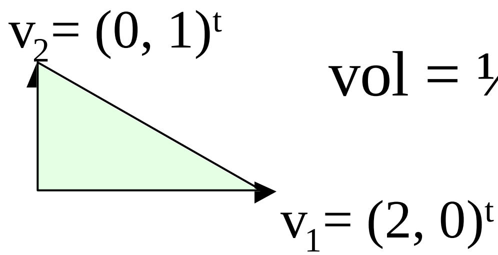
By inspection

vol(P) = vol(•) – vol(◥) = 4 – ½ = 3½

Simplicial Subdivision

Volume of one simplex:

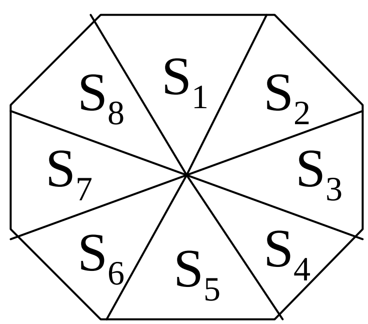
$$\mathrm{vol}(\mathrm{convh}\{0, v_1, \ldots, v_n\}) = \det[v_1 \ldots v_n]/n!$$

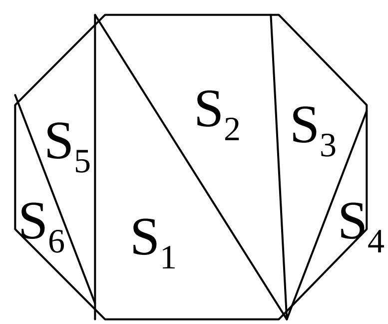


$$\mathrm{vol} = \tfrac{1}{2} \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$P = \cup S_i \text{ s.t. } \mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \varphi \text{ for } i \neq j$$

$$\Rightarrow \mathrm{vol}(P) = \Sigma_i \, \mathrm{vol}(S_i)$$





Other subdivisons

We can decompose $P = P_1 \cup P_2 \cup \ldots \cup P_k$ such that

$int(P_i) \cap int(P_j) = \varphi$ for $i \neq j$

so $vol(P) = vol(P_1) + vol(P_2) + \ldots + vol(P_k)$

- We know we can do it with simplexes, but we might not be able to "tile" P with other shapes (like squares).
- We could approximate vol(P) with *inner* and *outer* approximations that are easy to compute.

Mathematicians are like Frenchmen: whenever you say something to them, they translate it into their own language, and at once it is something entirely different.

J.W.v. Goethe

Approximations

- Inner - find vol(Q) for $Q \subseteq P$

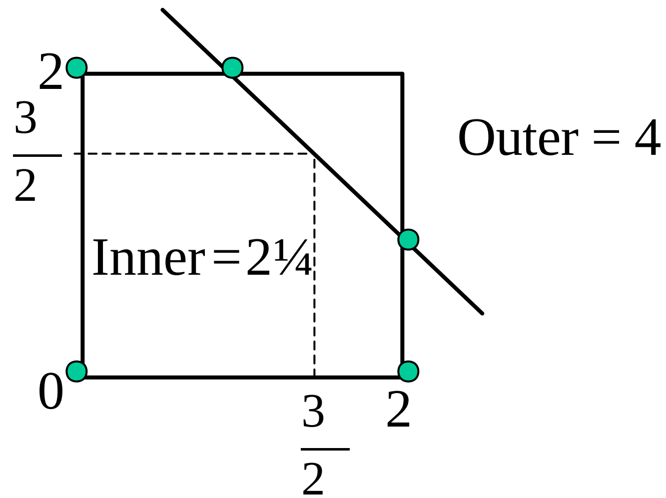
  $\max \Sigma_j \log x(j)$: $x \in P$, $x > 0$

  – easy convex program with linear constraints

- Outer - find vol(Q) for $Q \supseteq P$

  $\max x(j)$: $x \in P$

  – n LPs (or 2n LPs if min x(j) could be > 0)



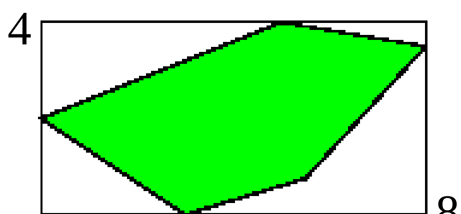
Monte Carlo

Solve $L_j = \min\{x_j : Ax \le b\}$ and $U_j = \max\{x_j : Ax \le b\}$

If $L_j = U_j$ for some j, eliminate x_j.

Now $L < U$ and assume P has full dimension.

Choose random number sequence and choose associated x in [L, U].
Let y(k) = # times $Ax \le b$ in k trials. Then,



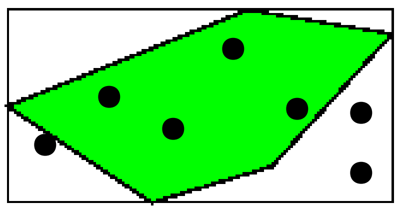
vol(Q) = 32

$$\text{vol}(P) = \text{vol}([L, U]) \lim_{k \to \infty} y(k)/k$$

$$= \prod_{j=1}^{n} (U_j - L_j)\, p$$

Extends using any Q for which vol(Q) is known, $P \subseteq Q$, and we can map random number into a point in Q.

Choosing random points



vol(P) est.= (4/7) ×32 = 18.29

for j=1:n

  r = pseudo random value in (0, 1)

  $x_j = L_j + r\times(U_j - L_j)$

end

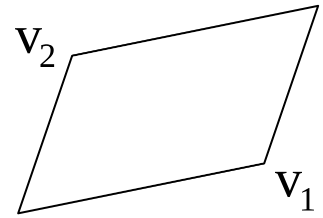
Turn to MATLAB code demo ☺

```
Count = 0;
for k=1:MaxIter
   x = rand(n,1).*U;
   if A*x <= b, Count = Count+1;
   end
end
p = Count/MaxIter
```

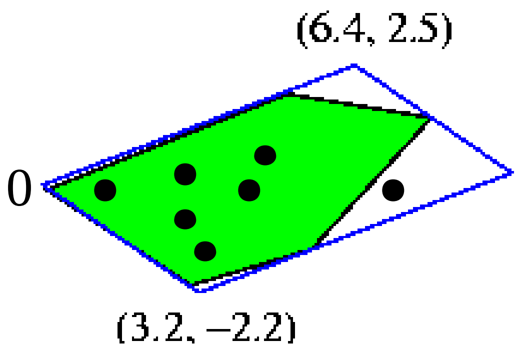
* Sharon Wiback had already implemented a version of this

Choosing tighter parallelepiped enclosure

$Q =$ parallelepiped $= \{\Sigma_j\, y_j v_j : 0 \le y_j \le 1\}$,
where $\{v_j\}$ are linearly independent



$$\mathrm{vol}(Q) = \left|\det[v_1 \ldots v_n]\right|$$



$$\mathrm{vol}(Q) = \det \begin{pmatrix} 6.4 & 3.2 \\ 2.5 & -2.2 \end{pmatrix} = 22.08$$

vol(P) est.= $(6/7) \times 22.08 = 18.93$

for j=1:n
  y_j = pseudo random value in (0, 1)
  $x_j = y_j \times v_j$
end

output: $x = \Sigma_j\, y_j v_j$ for $y \sim$ IIDU(0, 1)

Advantage of tighter Q

Given $P \subset Q \subset Q'$

Claim: Convergence of Monte Carlo is better using Q than Q'.

var(Count/k) = p(1–p)/k, where p = vol(P)/vol(Q),

so p near 1 or 0 is better than p near ½.

Near 0, however, has other problems with sample size.

Probabilists use the ratio **var:mean** as measure of convergence

→ minimizing 1–p is best.

i.e., Make vol(Q) as close to vol(P) as possible.

Computational note: vol(Q) and vol(P) could be very large, so either scale or estimate log vol(P) = log Count/k + log vol(Q).

Other areas of potential value

- Volonoi diagrams & Delaunay tesselations
- Sampling techniques
- Comparing polyhedra seems to be limited to 3D

Even if you're on the right track, you'll get run over if you just sit there.

Will Rogers

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